Hmwk: pg 176 #1, 2, 3,4-8(ace), 9, (10), (12), (16)

The Remainder Theorem

Review

Divide the following polynomials.

a) $\dfrac{x^3+2x^2-5x+7}{x-1}$ ← f(x)

$$\begin{array}{r|rrrr} 1 & 1 & 2 & -5 & 7 \\ & & 1 & 3 & -2 \\ \hline & 1 & 3 & -2 & 5 \end{array}$$

$= x^2+3x-2,\ R5$

b) $\dfrac{x^3+5x^2+2x-12}{x+3}$ ← f(x)

$$\begin{array}{r|rrrr} -3 & 1 & 5 & 2 & -12 \\ & & -3 & -6 & 12 \\ \hline & 1 & 2 & -4 & 0 \end{array}$$

$= x^2+2x-4,\ R\emptyset$

c) $\dfrac{2x^4-x^3+10x^2+4x+5}{x^2-x+5}$

$2x^2+x+1,\ R\emptyset$

$$\begin{array}{r} x^2-x+5\,\big)\,2x^4-x^3+10x^2+4x+5 \\ \underline{2x^4-2x^3+10x^2} \\ x^3+0x^2+4x \\ \underline{x^3-1x^2+5x} \\ x^2-1x+5 \\ \underline{x^2-1x+5} \\ 0 \end{array}$$

Consider examples a) and b) above. Suppose we substitute x = 1 in the numerator of a) and x = -3 in the numerator of b). What is the result?

a) $f(x) = x^3 + 2x^2 - 5x + 7$

$f(1) = (1)^3+2(1)^2-5(1)+7$

$= 1+2-5+7$

$= 5$

b) $f(x) = x^3 + 5x^2 + 2x - 12$

$f(-3) = (-3)^3+5(-3)^2+2(-3)-12$

$= -27+45-6-12$

$= 0$

The results are the same as the remainder from the first two examples above.

If we divide any polynomial f(x) by 'x - a', we get the following...

dividend → $\dfrac{f(x)}{x-a} = quotient,\ remainder$ ← divisor

dividend = (divisor)(quotient) + remainder

Rearranging, we get...

$f(x) = (x-a)(quotient) + remainder$

When we evaluate f(x) at the value x = a, we get...

$f(a) = (a-a)(quotient) + remainder$

$f(a) = remainder$

Remainder Theorem

When a polynomial f(x) is divided by bx - a, the remainder is equal to $f\left(\frac{a}{b}\right)$.

$b\left(\frac{a}{b}\right)-a$ ← if $x=\frac{a}{b}$

$= a-a$

$= 0$

Practice

Use the new theorem to determine the remainder of each quotient.

a) $\dfrac{\overbrace{x^4-2x^3+5x-4}^{f(x)}}{x-1}$

remainder $= f(1)$

$= (1)^4-2(1)^3+5(1)-4$

$= 1-2+5-4$

$= \emptyset$

b) $\dfrac{\overbrace{2x^3+3x^2+10}^{f(x)}}{x+2}$

remainder $= f(-2)$

$= 2(-2)^3+3(-2)^2+10$

$= -16+12+10$

$= 6$

c) $\dfrac{\overbrace{2x^2-11x+5}^{f(x)}}{2x-1}$

$2x-1=0$

$\frac{2x}{2}=\frac{1}{2}$

$x=\frac{1}{2}$

remainder $= f(\frac{1}{2})$

$= 2(\frac{1}{2})^2-11(\frac{1}{2})+5$

$= 2(\frac{1}{4})-\frac{11}{2}+5$

$= \frac{1}{2}-\frac{11}{2}+\frac{10}{2}$

$= \emptyset$

If the remainder of a quotient is zero, then the divisor is a factor of the dividend. Which divisors above are factors of the dividend? a) and c)

Factor Theorem

The binomial 'bx - a' is a factor of f(x), if and only if $f\left(\frac{a}{b}\right)$ is zero.

To factor a polynomial of degree 3 or larger, one or more factors are obtained by using a "guess and check" method with the "factor theorem".
**Hint: Choose integers that are factors of the last term of the polynomial. **

Example 1

Factor the following:

a) x^3-7x+6 $\quad$ $f(x)$ $\quad$ $\pm1, \pm2, \pm3, \pm6$

$f(1)=0$

$\therefore x-1$ is a factor

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -7 & 6 \\ & & 1 & 1 & -6 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$

$=(x-1)(x^2+x-6)$

$=(x-1)(x-2)(x+3)$

b) $\overbrace{x^3+5x^2-4x-20}^{f(x)}$

$f(1)=-18$

$f(-1)=-12$

$f(2)=\emptyset$ ✓

$\therefore x-2$ is a factor

$$\begin{array}{r|rrrr} 2 & 1 & 5 & -4 & -20 \\ & & 2 & 14 & 20 \\ \hline & 1 & 7 & 10 & 0 \end{array}$$

$=(x-2)(x^2+7x+10)$

$=(x-2)(x+2)(x+5)$

c) $\overbrace{x^4+8x^3+18x^2-27}^{f(x)}$

$f(1)=0$

$\therefore x-1$ is factor

$$\begin{array}{r|rrrrr} 1 & 1 & 8 & 18 & 0 & -27 \\ & & 1 & 9 & 27 & 27 \\ \hline & 1 & 9 & 27 & 27 & \emptyset \end{array}$$

$=(x-1)(\underbrace{x^3+9x^2+27x+27}_{g(x)})$

$g(-1)=8$

$g(-3)=\emptyset$ ✓

$\therefore x+3$ is a factor

$$\begin{array}{r|rrrr} -3 & 1 & 9 & 27 & 27 \\ & & -3 & -18 & -27 \\ \hline & 1 & 6 & 9 & \emptyset \end{array}$$

$=(x-1)(x+3)(x^2+6x+9)$

$=(x-1)(x+3)(x+3)(x+3)$

$=(x-1)(x+3)^3$

Example 2

Create a sketch of the following:

f(x) = 1x³ + 2x² - 4x - 8

$f(1) = -9$
$f(-1) = -3$
$f(2) = 0$
$\therefore$ x-2 is a factor

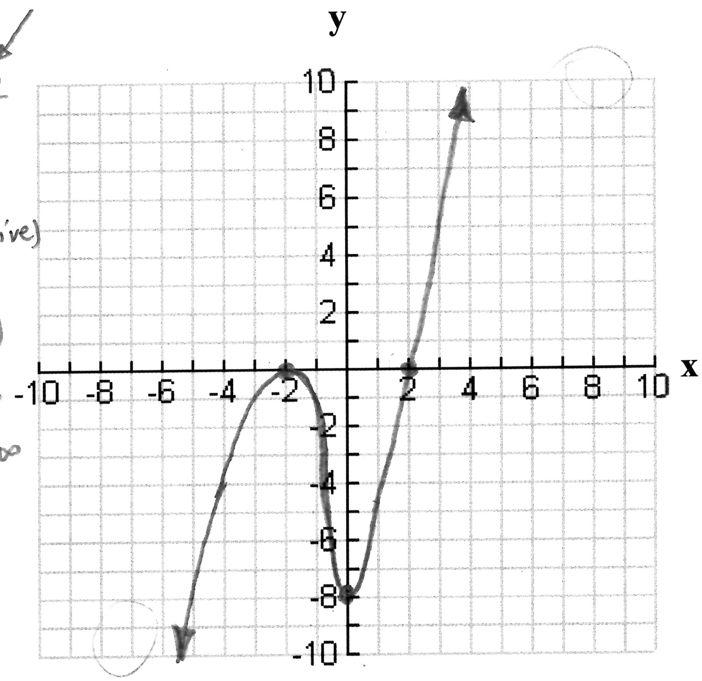
$$\begin{array}{r|rrrr} 2 & 1 & 2 & -4 & -8 \\ & & 2 & 8 & 8 \\ \hline & 1 & 4 & 4 & 0 \end{array}$$

$f(x) = (x-2)(x^2+4x+4)$
$= (x-2)(x+2)(x+2)$
$f(x) = (x-2)(x+2)^2$

turning point

x-int: 2 and -2
y-int: -8
lead. coeff. = 1 (positive)
degree = 3 (odd)

end behaviours:
as $x \to -\infty$, $y \to -\infty$
as $x \to \infty$, $y \to \infty$



In some special cases, we may be able to factor a high degree polynomial using grouping strategies; see below.

Example 3

Factor the following expressions by using a grouping strategy.

a) $x^4 - 6x^3 + 2x^2 - 12x$

$-\frac{6}{1} = -6$, $-\frac{12}{2} = -6$

$= x(x^3 - 6x^2 + 2x - 12)$
$= x[x^2(x-6) + 2(x-6)]$
$= x(x^2+2)(x-6)$

can't be factored
$x^2 + 0x + 2$
$b^2 - 4ac = (0)^2 - 4(1)(2)$
$= -8$ negative

b) $2x^3 + 4x^2 + 3x + 6$

$\frac{4}{2} = 2$, $\frac{6}{3} = 2$

$= 2x^2(x+2) + 3(x+2)$
$= (2x^2+3)(x+2)$

not factorable