Binomial Theorem

Pascal's Triangle	Binomial Expansion
row 0 1	$(a+b)^0 = 1$
row 1 1 1	$(a+b)^1 = 1a + 1b$
row 2 1 2 1	$(a+b)^2 = 1a^2 + 2ab + 1b^2$
row 3 1 3 3 1	$(a+b)^3 = 1a^3 + 3a^2b + 3a^1b^2 + 1b^3$
row 4 1 4 6 4 1	$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4a^1b^3 + 1b^4$

* done on third page

What do you notice about the expansions of $(a + b)^n$?

- The simplified expansions have n+1 terms.
- The exponents of 'a' start at a value 'n' and decreases by one for each successive term until it is reduced to 0.
- The exponents of 'b' start at a value 0 and increases by one for each successive term until it reaches 'n'.
- The coefficients in the simplified expansion are identical to the elements of Pascal's Triangle in row n; the first and last coefficient is always 1.

Additionally,

$$\begin{array}{c} \text{coefficient of} \\ \text{the next term} \end{array} = \frac{\text{current coefficient} \times \text{current exponent of 'a'}}{\text{current term number}}$$

Examples

* Careful

Use Pascal's triangle to expand the following binomials:

a) $(x + 2y)^4 = 1(x)^4(2y)^0 + 4(x)^3(2y)^1 + 6(x)^2(2y)^2 + 4(x)^1(2y)^3 + 1(x)^0(2y)^4$

a b

$= 1(x^4)(1) + 4(x^3)(2y) + 6(x^2)(4y^2) + 4(x)(8y^3) + 1(1)(16y^4)$

$= x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4$

b) $(-3x + 2y)^3 = 1(-3x)^3(2y)^0 + 3(-3x)^2(2y)^1 + 3(-3x)^1(2y)^2 + 1(-3x)^0(2y)^3$

a b

$= 1(-27x^3)(1) + 3(9x^2)(2y) + 3(-3x)(4y^2) + 1(1)(8y^3)$

$= -27x^3 + 54x^2y - 36xy^2 + 8y^3$

$+ + + +$
$+ - + -$
$- + - +$ ✓
$- - - -$

c) $\left(\frac{1}{2} - 3y\right)^{3} = 1\left(\frac{1}{2}\right)^3(-3y)^0 + 3\left(\frac{1}{2}\right)^2(-3y)^1 + 3\left(\frac{1}{2}\right)^1(-3y)^2 + 1\left(\frac{1}{2}\right)^0(-3y)^3$

a b

$= 1\left(\frac{1}{8}\right)(1) + \frac{3}{1}\left(\frac{1}{4}\right)\left(\frac{-3y}{1}\right) + \frac{3}{1}\left(\frac{1}{2}\right)\left(\frac{9y^2}{1}\right) + 1(1)(-27y^3)$

$= \frac{1}{8} - \frac{9}{4}y + \frac{27}{2}y^2 - 27y^3$

Homework: Expand the following.

a) $\cancel{(1 + 3x)^3}$ $(x + 3y)^3$ b) $(1 - x)^4$ c) $\cancel{(1 - 5x)^5}$ $(x - 5y)^3$ d) $\left(x - \frac{1}{2}\right)^{\cancel{6}5}$

\+ pg 466 #1, 2ab, 3ab, 4ab, 5ab, 6a, 10

$$(a+b)^3 = (a+b)(a+b)(a+b)$$
$$= (a^2+ab+ab+b^2)(a+b)$$
$$= \cancel{a^3}+\cancel{a^2b}+a^2b+\cancel{ab^2}+a^2b+\cancel{ab^2}+ab^2+b^3$$
$$= a^3+3a^2b+3ab^2+b^3$$

$$(a+b)^4 = (a+b)(a+b)(a+b)(a+b)$$
$$= (a+b)(a^3+3a^2b+3ab^2+b^3)$$
$$= \cancel{a^4}+\cancel{3a^3b}+\cancel{3a^2b^2}+ab^3+a^3b+3a^2b^2+\cancel{3ab^3}+b^4$$
$$= a^4+4a^3b+6a^2b^2+4ab^3+b^4$$

Homework

a) $(x+3y)^3 = 1(x)^3(3y)^0+3(x)^2(3y)^1+3(x)^1(3y)^2+1(x)^0(3y)^3$

a b

$$= 1(x^3)(1)+3(x^2)(3y)+3(x)(9y^2)+1(1)(27y^3)$$
$$= x^3+9x^2y+27xy^2+27y^3$$

b) $(1-x)^4 = 1(1)^4(-x)^0+4(1)^3(-x)^1+6(1)^2(-x)^2+4(1)^1(-x)^3+1(1)^0(-x)^4$

a b

$$= 1(1)(1)+4(1)(-x)+6(1)(x^2)+4(1)(-x^3)+1(1)(x^4)$$
$$= 1-4x+6x^2-4x^3+x^4$$

c) $(x-5y)^3 = 1(x)^3(-5y)^0+3(x)^2(-5y)^1+3(x)^1(-5y)^2+1(x)^0(-5y)^3$

a b

$$= 1(x^3)(1)+3(x^2)(-5y)+3(x)(25y^2)+1(1)(-125y^3)$$
$$= x^3-15x^2y+75xy^2-125y^3$$

d) $(x-\frac{1}{2})^5 = 1(x)^5(-\frac{1}{2})^0+5(x)^4(-\frac{1}{2})^1+10(x)^3(-\frac{1}{2})^2+10(x)^2(-\frac{1}{2})^3+5(x)^1(-\frac{1}{2})^4+1(x)^0(-\frac{1}{2})^5$

$$= 1(x^5)(1)+5(x^4)(-\tfrac{1}{2})+10(x^3)(\tfrac{1}{4})+10(x^2)(-\tfrac{1}{8})+5(x)(\tfrac{1}{16})+1(1)(-\tfrac{1}{32})$$
$$= x^5-\frac{5}{2}x^4+\frac{5}{2}x^3-\frac{5}{4}x^2+\frac{5}{16}x-\frac{1}{32}$$